Bose had originally submitted his pioneering paper on what later became known as Bose-Einstein statistics to the *Philosophical Magazine*, but the manuscript was rejected there. He then sent it to Einstein, who accepted it for publication in the *Zeitschrift für Physik* and personally translated it from English to German. The following is a back-translation of the Einstein translation of Bose's work, as it finally appeared in *Z. Phys.* **26**, pp. 178–181 (1924). An attempt has been made to keep this back-translation as close as possible to the German "original", both in style and in notation.

Planck's law and the hypothesis of light quanta

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(Received 2 July 1924)

The phase space of a quantum of light with respect to a given volume is divided into "cells" of magnitude h^3. The number of possible distributions of the light quanta of macroscopically defined radiation over these cells yields the entropy and thus all thermodynamic properties of the radiation.

Planck's formula for the distribution of energy in the radiation of a black body constitutes the starting point for quantum theory, which has been developed during the last 20 years and borne rich fruits in all fields of physics. Since its publication in 1901, many derivations of this law have been suggested. It is accepted that the fundamental propositions of quantum theory can not be reconciled with the laws of classical electrodynamics. All previous derivations utilize the relation

$$\rho_\nu \mathrm{d}\nu = \frac{8\pi\nu^2 \mathrm{d}\nu}{c^3} E \ ,$$

i.e., the relation between the density of radiation and the mean energy of an oscillator, and they make assumptions concerning the number of degrees of freedom of the ether, as it enters into the above equation (first factor on the right hand side). However, this factor could only be deduced from the classical theory. This is the unsatisfactory point in all derivations, and it is not surprising that repeatedly serious attempts have been made to give a derivation which is free of this logical error.

A remarkably elegant derivation has been presented by *Einstein*. This author has recognized the logical deficiency of all previous derivations and attempted to deduce the formula without reference to the classical theory. Starting from very elementary assumptions concerning the exchange of energy between molecules and the radiation field, he finds the relation

$$\rho_\nu = \frac{\alpha_{mn}}{e^{\frac{\varepsilon_m - \varepsilon_n}{kT}} - 1} \ .$$

However, in order to bring this formula into agreement with the one by *Planck*, he has to make use of *Wien*'s displacement law and *Bohr*'s correspondence principle. *Wien*'s law is based on the classical theory, and the correspondence principle supposes that the quantum theory be in agreement with the classical theory in certain limits.

In all cases it seems to me that the derivations are not justified in a sufficiently logical manner. On the other hand, it seems to me that the hypothesis of light quanta in connection with statistical mechanics (as it has been adapted to the demands of quantum theory by *Planck*) is sufficient for the derivation of the law without taking recourse to the classical theory. In the following I will briefly sketch the method.

Let the radiation be enclosed in a volume V, and let its total energy be E. Let there be different kinds of quanta with the respective numbers N_s and energies $h\nu_s$ ($s = 0$ to $s = \infty$). The total energy E then is

$$E = \sum_s N_s h\nu_s = V \int \rho_\nu \, \mathrm{d}\nu \ . \tag{1}$$

The solution of the problem then requires the calculation of the quantities N_s, which determine ρ_ν. If we know the probability for each arbitrary distribution characterized by N_s, then the solution is determined by the condition that this probability, while respecting the constraint (1), be a maximum. We now want to find this probability.

The quantum has a momentum of magnitude $\frac{h\nu_s}{c}$ in the direction of its movement. The instantaneous state of the quantum is characterized by its coordinates x, y, z and the conjugate momenta p_x, p_y, p_z; these six quantities can be interpreted as point coordinates in a six-dimensional space, obeying the relation

$$p_x^2 + p_y^2 + p_z^2 = \frac{h^2\nu^2}{c^2} \ ,$$

by means of which the aforementioned point is forced to stay on the surface of a cylinder determined by the quantum's frequency. In this sense, the frequency interval $\mathrm{d}\nu_s$ corresponds to the phase space

$$\int \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z\ \mathrm{d}p_x\ \mathrm{d}p_y\ \mathrm{d}p_z = V \cdot 4\pi \left(\frac{h\nu}{c}\right)^2 \frac{h\mathrm{d}\nu}{c} = 4\pi \frac{h^3\nu^2}{c^3} V \mathrm{d}\nu \ .$$

If we divide the total phase space volume into cells of magnitude h^3, the frequency interval $\mathrm{d}\nu$ thus corresponds to $4\pi V \frac{\nu^2}{c^3}\mathrm{d}\nu$ cells. As regards the nature of this division, nothing definite can be said. However, the total number of cells has to be interpreted as the number of possible arrangements of a quantum in the given volume. To account for the fact of polarization it seems mandatory to further multiply this number by 2, so that we obtain $8\pi V \frac{\nu^2}{c^3}\mathrm{d}\nu$ for the number of cells belonging to $\mathrm{d}\nu$.

Now it is a simple task to calculate the thermodynamic probability of a (macroscopically defined) state. Let N^s be the number of quanta belonging to the frequency interval $\mathrm{d}\nu^s$. How many ways are there to distribute those over the cells belonging to $\mathrm{d}\nu^s$? Let p_0^s be the number of empty cells, p_1^s the number of those which contain one quantum, p_2^s the number of cells containing two quanta, and so on. The number of possible distributions then is

$$\frac{A^s!}{p_0^s!\, p_1^s! \ \ldots} \ , \quad \text{where} \ \ A^s = \frac{8\pi\nu^2}{c^3}\mathrm{d}\nu^s \ ,$$

and where

$$N^s = 0 \cdot p_0^s + 1 \cdot p_1^s + 2 \cdot p_2^s \ \ldots$$

is the number of quanta belonging to $\mathrm{d}\nu^s$.

The probability of the state defined by all p_r^s obviously is

$$\prod_s \frac{A^s!}{p_0^s!\, p_1^s! \ \ldots} \ .$$

Accounting for the fact that we may consider the p_s^r as large numbers, we have

$$\ln W = \sum_s A^s \ln A^s - \sum_s \sum_r p_r^s \ln p_r^s \ ,$$

where

$$A^s = \sum_r p_r^s \ .$$

This expression is required to be a maximum, subject to the constraint

$$E = \sum_s N^s h\nu^s \ ; \quad N^s = \sum_r r p_r^s \ .$$

Performing the variation yields the conditions

$$\sum_s \sum_r \delta p_r^s \left(1 + \ln p_r^s\right) = 0 \; , \qquad \sum_s \delta N^s h\nu^s = 0$$

$$\sum_r \delta p_r^s = 0 \; , \qquad \delta N^s = \sum_r \delta p_r^s \; .$$

Hence follows

$$\sum_s \sum_r \delta p_r^s \left(1 + \ln p_r^s + \lambda^s\right) + \frac{1}{\beta} \sum_s h\nu^s \sum_r r \delta p_r^s = 0 \; .$$

From this follows immediately

$$p_r^s = B^s e^{-\frac{rh\nu^s}{\beta}} \; .$$

However, since

$$A^s = \sum_r B^s e^{-\frac{rh\nu^s}{\beta}} = B^s \left(1 - e^{-\frac{h\nu^s}{\beta}}\right)^{-1} \; ,$$

we obtain

$$B^s = A^s \left(1 - e^{-\frac{h\nu^s}{\beta}}\right) \; .$$

Moreover, one has

$$\begin{aligned} N^s = \sum_r r p_r^s &= \sum_r r A^s \left(1 - e^{-\frac{h\nu^s}{\beta}}\right) e^{-\frac{rh\nu^s}{\beta}} \\ &= \frac{A^s e^{-\frac{h\nu^s}{\beta}}}{1 - e^{-\frac{h\nu^s}{\beta}}} \; . \end{aligned}$$

Thus, invoking the value of A^s found above,

$$E = \sum_s \frac{8\pi h \nu^{s3} \mathrm{d}\nu^s}{c^3} V \frac{e^{-\frac{h\nu^s}{\beta}}}{1 - e^{-\frac{h\nu^s}{\beta}}} \; .$$

Using the previous results one also finds

$$S = k \left[\frac{E}{\beta} - \sum_s A^s \ln \left(1 - e^{-\frac{h\nu^s}{\beta}}\right) \right] \; ,$$

which, taking into account $\frac{\partial S}{\partial E} = \frac{1}{T}$, yields $\beta = kT$. Inserting this into the above equation for E, one arrives at

$$E = \sum_s \frac{8\pi h \nu^{s3}}{c^3} V \frac{1}{e^{\frac{h\nu^s}{kT}} - 1} \mathrm{d}\nu^s \; ,$$

which equation is equivalent to *Planck*'s formula.

(Translated by *A. Einstein*.)

Note added by the translator. Bose's derivation of *Planck*'s formula constitutes, in my opinion, an important step forward. The method used here also yields the quantum theory of the ideal gas, as I will explain elsewhere.