Analog of photon-assisted tunneling in a Bose–Einstein condensate

André Eckardt, Tharanga Jinasundera, Christoph Weiss, and Martin Holthaus
Institut für Physik, Carl von Ossietzky Universität, D-26111 Oldenburg, Germany
(Dated: July 21, 2005)

We study many-body tunneling of a small Bose–Einstein condensate in a periodically modulated, tilted double-well potential. Periodic modulation of the trapping potential leads to an analog of photon-assisted tunneling, with distinct signatures of the interparticle interaction visible in the amount of particles transferred from one well to the other. In particular, under experimentally accessible conditions there exist well-developed half-integer Shapiro-like resonances.

PACS numbers: 03.75.Lm, 74.50.+r

In a recent experiment, it has become possible to monitor the Josephson-like oscillation of a sample of about 1000 Bose–Einstein-condensed atoms in an optical double-well potential, and to observe in situ both the evolution of the atomic densities, and of the relative phase between the condensates in both wells [1]. In the present Letter, we propose to extend such experiments by modulating the trapping potential periodically in time. As we will demonstrate, such modulated double-well condensates exhibit certain parallels to photon-assisted tunneling in microwave-driven superconducting Josephson junctions [2], although the “photons” now tend to fall into the lower Kilohertz regime.

We consider a symmetric double-well trap filled with N Bose particles at temperature T = 0, as realized approximately in the Heidelberg experiment [1]. Denoting the tunneling splitting between the lowest pair of single-particle energy eigenstates by ℏΩ, and adopting the common two-mode approximation [3–5], the system’s Hamiltonian takes the idealized form

\[ H_0 = -\frac{\hbar\Omega}{2} (a_i^+ a_{i+1} - a_i a_{i+1} + a_i^+ a_{i+2} - a_i a_{i+2}) + \hbar \kappa (a_i^+ a_i^+ a_i a_i + a_i a_i^+ a_i^+ a_i) , \tag{1} \]

where \( a_i^+ \) (\( a_i \)) is the annihilation (creation) operator for a Boson in the \( i \)-th well, satisfying the commutation relation \([a_i, a_j^+] = \delta_{i,j} (i, j = 1, 2)\). The on-site interaction energy of a single pair of Bosons occupying the same well is \( 2\hbar a_s \), proportional to the s-wave scattering length of the particle species. This deceptively simple model (1), describing an unforced Bosonic Josephson junction, has recently been studied in considerable detail [6, 7]. The key parameter governing its dynamics is the dimensionless ratio \( N\kappa/\Omega \). While the well-understood mean-field approximation [8–10] corresponds to the limit \( N \rightarrow \infty \) and \( \kappa \rightarrow 0 \), taken such that the product \( N\kappa \) remains constant, present experiments start to explore the dynamics beyond the mean-field regime [1].

We extend this system by introducing a twofold bias: We assume that the two wells are tilted such that their bottoms are misaligned in energy by an amount \( 2\hbar\mu_0 \), which should be sufficiently large compared to the tunnel splitting \( \hbar\Omega \), so that the usual Josephson oscillations are strongly suppressed. In addition, we propose to modulate this tilted double well periodically in time, such that the individual wells are shifted sinusoidally up and down, in phase opposition to each other and with angular frequency \( \omega \), by an amount \( \hbar\mu_1 \). Such a modulation of an optical double well can be achieved by periodically shifting the focus of a blue-detuned laser, which creates the barrier between the two wells, with the help of a piezo-actuated mirror [1]. Thus, we are led to the explicitly time-dependent many-body Hamiltonian

\[ H(t) = H_0 + h(\mu_0 + \mu_1 \sin \omega t) (a_i^+ a_{i+1} - a_i a_{i+1} + a_i^+ a_{i+2} - a_i a_{i+2}) . \tag{2} \]

As follows from elementary estimates [3, 4, 11], the unperturbed two-mode Hamiltonian (1) can be trusted as long as the particle number \( N \) does not exceed the ratio of the characteristic length scale \( L \) of one of the wells and the scattering length \( a_s \); for \( \mu \)-sized traps filled with alkali atoms, this ratio \( L/a_s \) will typically be on the order of 1000. However, the ability to reduce atomic scattering lengths by means of Feshbach resonances [12] allows one to extend the two-mode approximation to much larger samples. Moreover, both Rabi frequencies \( \mu_0 \) and \( \mu_1 \) and the modulation frequency \( \omega \) should remain restricted to a few times the tunneling frequency \( \Omega \). With tunneling times on the order of some 100 ms [1, 11], this leads to modulation frequencies in the low kHz-regime.

In the following, we investigate the dynamics of the model (2). In our calculations we assume that initially, at time \( t = 0 \), all \( N \) Bose particles are prepared in the lower well (that is, in the well tilted downward by \(-\hbar\mu_0\)). When solving the time-dependent Schrödinger equation numerically, we record the expectation value

\[ \langle J_z \rangle(t) = \frac{1}{2} \langle \psi(t) | a_i^+ a_{i+1} - a_i a_{i+1} | \psi(t) \rangle \tag{3} \]

which quantifies the imbalance of the numbers of particles found in the individual wells in the course of time, such that \( \langle J_z \rangle(t_0)/N = +1/2 (-1/2) \) means that all particles occupy the lower (higher) well at \( t_0 \). We keep the static tilt fixed at three times the tunneling splitting, \( 2\hbar\mu_0 = 3\hbar\Omega \). As shown in Fig. 1, this value allows merely a small-amplitude oscillation of the population imbalance when there is no periodic modulation (dashed line).
FIG. 1: Time evolution of the population imbalance $\langle J_z \rangle / N$ for a Bosonic Josephson junction with $N = 1000$ particles, interaction parameter $N\kappa/\Omega = 0.5$, and tilt $2\mu_0/\Omega = 3.0$. The dashed line refers to the system without periodic forcing; the two full lines are obtained for periodic modulation with one-photon resonant frequency $\omega/\Omega = 3.0$. The strongly oscillating line corresponds to the driving amplitude $2\mu_1/\omega = 1.8$, close to the first maximum of the Bessel function $J_1$ (inset: solution to the nonlinear Schrödinger equation for this situation), while the full line at the top of the figure results for $2\mu_1/\omega = J_{1,1}$, the first zero of $J_1$.

In order to condense information about the system’s response to bias with various parameters in a single viewpoint, we characterize each quantum trajectory by its time-averaged imbalance

$$\langle J_z \rangle_t \equiv \frac{1}{\Delta t} \int_0^{\Delta t} dt \, \langle J_z \rangle(t),$$

employing the averaging interval $\Delta t = 100/\Omega$. Figure 2 depicts results of such calculations for a periodic modulation with fixed scaled amplitude $2\mu_1/\omega = 0.5$, and frequencies $\omega$ ranging from $0.5\Omega$ to $6\Omega$. By construction, values of $\langle J_z \rangle_t / N$ close to 0.5 indicate that the condensate remains trapped almost entirely in the initially occupied well during $\Delta t$, whereas values close to zero signal that it visits both wells about equally. From the physics of superconducting Josephson junctions, one expects photon-assisted tunneling to take place when the energy of an integer number of photons matches the static tilt, $n\hbar\omega = 2h\mu_0$, as in the case of Shapiro resonances [2]. Figure 2 indeed reveals a pronounced single-photon resonance at $\omega/\Omega \approx 3.0$, clear signatures of two-photon-assisted tunneling at $\omega/\Omega \approx 1.5$, and even traces of a three-photon process at $\omega/\Omega \approx 1.0$. However, the actual resonance condition does not refer directly to the tilt $2h\mu_0$, but rather to the tunneling frequency of the condensate without periodic forcing: Within the mean-field approximation, the dynamical system (2) corresponds to a driven nonlinear pendulum [8]; in such a system, a resonance can occur when a rational multiple of the modulation frequency $\omega$ matches the oscillation frequency $\omega_0$ of the unforced pendulum [13]. This resonance condition obviously is effective even when there is no atom-atom interaction. For $N\kappa/\Omega = 0$, one has $\omega_0(0)^2 = 4\mu_0^2 + \Omega^2$, shifting the one-photon resonance from the Shapiro value $\omega/\Omega = 2\mu_0/\Omega = 3$ to $\omega/\Omega \approx 3.162$. Even with weak interaction, $N\kappa/\Omega = 0.1$, this shift remains visible in the location of the main resonance peak in Fig. 2.

With increasing interaction $N\kappa/\Omega$, the Shapiro-like resonances depicted in Fig. 2 shift downwards in frequency. (They shift upwards, by different amounts, if initially the higher well is occupied.) The mean-field approximation allows one to explain this observation quantitatively by expressing the oscillation frequency $\omega_0(N\kappa/\Omega)$ of the undriven, interacting system in terms of elliptic integrals [10]. We have checked that the condition $n\omega \approx n\omega_0(N\kappa/\Omega)$ with $n = 1, 2, 3$ reproduces the locations of the one-, two-, and three-photon-like resonances quite well.

Figure 2 also reveals a well-developed half-integer resonance at $\omega/2 \approx \omega_0(N\kappa/\Omega)$, corresponding to $n = 1/2$. Such subharmonic resonances can again be understood with the help of the nonlinear Schrödinger equation, which describes the dynamics of the driven condensate within the mean-field approximation: The nonlinearity effectively generates both harmonics and subharmonics of the driving frequency. For a matter-of-principle demonstration, Fig. 3 shows the time-averaged imbalance provided by the nonlinear Schrödinger equation for stronger forcing with scaled amplitude $2\mu_1/\omega = 1.8$ and rather weak interaction $N\kappa/\Omega = 0.1$, together with the corresponding data for a non-interacting system subjected to the same forcing. While the mean-field result exhibits half-integer resonances with $n = 3/2$ and $n = 5/2$, these resonances vanish when $N\kappa/\Omega = 0$. On the other hand,
the graph of the imbalance (4) provided by the \(N\)-particle Schrödinger equation for \(N = 1000\) (not shown) is practically indistinguishable from that of the mean-field data displayed in Fig. 3, and even a two-particle system yields a pronounced peak for \(n = 3/2\) when \(N \kappa/\Omega\) is adjusted to 0.1. In general, while the time-dependent \(N\)-particle imbalance (3) follows the mean-field data only in the limit \(N \to \infty\) (with \(\kappa\) adjusted such that \(N \kappa\) stays constant), but soon deviates from its mean-field counterpart when \(N\) is of the order of 1000 (cf. Fig. 1), it is remarkable that its average (4) comes fairly close to the mean-field approximation even when \(N\) is much smaller.

A powerful theoretical tool for studying the many-body dynamics of the driven Bosonic Josephson junction is provided by quantum Floquet theory: Since the Hamiltonian (2) is periodic in time, \(H(t) = H(t + T)\) with period \(T = 2\pi/\omega\), there exists a complete set of solutions to the time-dependent Schrödinger equation of the form \(|\psi_m(t)\rangle = |u_m(t)\rangle \exp(-i\varepsilon_m t/\hbar)\), with the Floquet functions \(|u_m(t)\rangle = |u_m(t + T)\rangle\) sharing the \(T\)-periodicity of the Hamiltonian [14]. The quantities \(\varepsilon_m\), which serve to describe the time evolution in a similar manner as energy eigenvalues do in the case of energy eigenstates, are called quasienergies. The existence of these time-dependent many-body states, which incorporate both the interparticle interactions and the external forcing in a non-perturbative manner, is not tied to the particular model Hamiltonian (2), but stems solely from the temporal periodicity of the forcing, exactly as the existence of Bloch states in solid state physics is solely due to the spatial periodicity of a crystalline lattice [15].

While the exact calculation of Floquet states and their quasienergies in general has to resort to numerical means [11], the particular case of an integer resonance, \(n\omega = 2\mu_0\), allows for an instructive analytical approximation. Provided the static tilt is sufficiently large to substantially reduce the unperturbed Josephson oscillations, and assuming low resonance order \(n\), so that \(\omega/\Omega\) remains large compared to \(N \kappa/\Omega\), the quasienergies of the driven junction (2) are approximately given by the energy eigenvalues of the undriven junction (1), but with an effective tunneling frequency

\[
\Omega_{\text{eff}} = \Omega J_n\left(\frac{2\mu_1}{\omega}\right),
\]

where \(J_n(z)\) is an ordinary Bessel function of order \(n\). This renormalization of the tunneling frequency under resonant driving resembles the renormalization of atomic \(g\)-factors by radiofrequency fields [16], or the coherent destruction of single-particle tunneling [17].

To demonstrate that this concept is meaningful under experimentally accessible conditions, we have displayed in Fig. 1 the evolution of the population imbalance (3) under the influence of a strong modulation: The lower, strongly oscillating full line results from one-photon resonant driving with scaled amplitude \(2\mu_1/\omega = 1.8\), which is close to the first maximum of \(J_1\). Correspondingly, one observes photon-assisted tunneling with almost complete population exchange during the initial stage, leveling down to smaller-amplitude oscillations which average to zero. As remarked above, the solution to the nonlinear Schrödinger equation for the same situation (inset) deviates strongly from the \(N\)-particle solution after only a few periods. The upper full line in Fig. 1 results from one-photon resonant driving with \(2\mu_1/\omega \approx 3.832\), the first zero of \(J_1\): Since the effective tunneling frequency (5) vanishes then, the particles remain caught almost perfectly in the initially occupied well.

The renormalization (5) prompts at an interplay between photon-assisted tunneling and an effect known as
self-trapping. It has been established that when the absolute value of the interaction parameter $|N\kappa/\Omega|$ exceeds a certain critical value, which equals 1 for maximum initial population imbalance, the nonlinear Josephson oscillations occurring in the undriven junction become suppressed [8, 10]; this self-trapping phenomenon has by now been observed in the pioneering experiment [1]. Therefore, employing driving frequencies sufficiently high for the modulation-induced renormalization (5) to hold, and starting from an initial state with all particles in one well, we expect self-trapping to occur in a resonantly driven Bosonic Josephson junction when

$$\frac{N\kappa}{\Omega J_0(2\mu_1/\omega)} > 1.$$  \hspace{1cm} (6)

This expectation is confirmed in Fig. 4, where we have plotted time-averaged population imbalances as functions of the scaled driving amplitude, again for the example of a one-photon resonance. The number of particles has been reduced to $N = 100$ here: As far as the averages (4) are concerned, this yields practically the same result as $N = 1000$, and even as the nonlinear Schrödinger equation, despite the differences illustrated in Fig. 1. For very low driving amplitudes, $2\mu_1/\omega < 0.1$, the forcing is hardly effective, so that the static tilt keeps the particles in the initially occupied well. With increasing amplitude, photon-assisted tunneling sets in. For $2\mu_1/\omega$ larger than about 1.5, the four curves drawn in Fig. 4 behave rather differently: For low interaction parameter $N\kappa/\Omega = 0.1$ (full line), the trapping condition (6) is effective only in comparatively small neighborhoods of the zeros $j_{1,1} \simeq 3.832, j_{2,1} \simeq 7.016$, and $j_{1,3} \simeq 10.173$ of $J_1$. When $N\kappa/\Omega$ is increased to 0.3 (short dashes), these neighborhoods become wider; in the entire interval between $j_{1,2}$ and $j_{1,3}$ the condition (6) is met, so that the imbalance does not approach zero. For $N\kappa/\Omega = 0.5$ (long dashes) self-trapping occurs already between the first and second zero, while $N\kappa/\Omega = 0.7$ (dots) does not admit full population exchange even for scaled amplitudes below $j_{1,1}$, in agreement with the inequality (6). Thus, the interactions among the Bose particles exert a strong influence on photon-assisted tunneling, leading to a distinctly non-monotonous dependence of the population transfer on the driving amplitude, and allowing substantial transfer only in limited subsets of the parameter space.

In conclusion, we have suggested to extend the ongoing experimental investigation of condensate tunneling in optical double wells [1] by a time-periodic modulation of the trapping potential, thereby opening up parallels to the ac Josephson effect. Many-body Floquet theory leads to the prediction of the amplitude-dependent breathing (5) of the well-to-well coupling under strong resonant driving. However, driven Bosonic Josephson junctions offer further peculiarities of their own, such as interaction-induced resonance shifts, and clearly developed half-integer Shapiro-like resonances, which should be verifiable under presently accessible laboratory conditions. Subharmonic Shapiro steps in microwave-driven superconducting point contacts have recently been ascribed to multiple Andreev reflections [18]. The fact that related phenomena occur in modulated optical Bosonic junctions, where all of the parameters $N, \Omega, \kappa, \mu_0, \mu_1$, and $\omega$ can be tuned separately, opens up far-reaching new avenues of investigation.

This work was supported by the Deutsche Forschungsgemeinschaft through the Priority Programme SPP 1116, Wechselwirkung in ultrakalten Atom- und Molekülgasen. A.E. acknowledges a fellowship from the Studienstiftung des deutschen Volkes.